e content for students of patliputra university

B. Sc. (Honrs) Part 1paper 1

Subject:Mathematics

Topic:symmetric function of the roots

symmetric function of the roots

If a function involving all the roots of an equation is unaltered : the roots are interchanged, it is called a symmetric function of the roots.

Let $\alpha_1, \alpha_2, \alpha_3, \ldots\ldots\ldots \alpha_n$ be the roots of the equation.

$$f(x) = x^n + p_1 x^{n-1} + p_2 x^{n-2} + \ldots. + p_n = 0.$$

We have :

$$S_1 = \Sigma\, \alpha_1 = -\, p_1$$

$$S_2 = \Sigma\, \alpha_1 \alpha_2 = p_2$$

$$S_3 = \Sigma\, \alpha_1\, \alpha_2\, \alpha_3 = -\, p_3$$

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Without knowing the values of the roots separately in terms of the coefficients, by using the above relations between the coefficients and the roots of an equation, we can express any symmetric function of the roots in terms of the coefficients of the equations.

Example 1. If α, β, γ are the roots of the equations $x^3 + px^2 + qx + r = 0$, Express the value of $\Sigma\, \alpha^2\, \beta$ in terms of the coefficients.

Solution.

We have $\alpha + \beta + \gamma = -\, p$

$$\alpha\beta + \beta\gamma + \gamma\alpha = q$$

$$\alpha\beta\gamma = -\, r.$$

$$\Sigma\, \alpha^2\beta = \alpha^2\beta + \alpha^2\gamma + \beta^2\alpha + \beta^2\gamma + \gamma^2\alpha + \gamma^2\beta$$

$$= (\alpha\beta + \beta\gamma + \gamma\alpha)\, (\alpha + \beta + \gamma) - 3\alpha\beta\gamma$$

$$= q\, (-\, p) - 3\, (-\, r)$$

$$= 3r - pq.$$

Example 2. If α, β, γ, δ be the roots of the bi quadratic equation $x^4 + px^3 + qx^2 + rx + s = 0$, Find (1) $\Sigma\, \alpha^2$, (2) $\Sigma\, \alpha^2\, \beta\gamma$, (3) $\Sigma\, \alpha^2\, \beta^2$, (4) $\Sigma\, \alpha^3\, \beta$ and (5) $\Sigma\, \alpha^4$.

Solution.

The relations between the roots and the coefficients are

$$\alpha + \beta + \gamma + \delta = -\, p.$$

$$\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = q$$

$$\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = -\, r$$

$$\alpha\beta\gamma\delta = s.$$

$$\Sigma \alpha^2 = \alpha^2 + \beta^2 + \gamma^2 + \delta^2$$

$$= (\alpha + \beta + \gamma + \delta)^2 - 2 (\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta)$$

$$= (\Sigma \alpha)^2 - 2 \Sigma \alpha\beta$$

$$= p^2 - 2q.$$

$$\Sigma \alpha^2 \beta\gamma = (\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta) (\alpha + \beta + \gamma + \delta) - 4 \alpha\beta\gamma\delta$$

$$= (\Sigma \alpha\beta\gamma) (\Sigma \alpha) - 4 \alpha\beta\gamma\delta$$

$$= pr - 4s.$$

$$\Sigma \alpha^2 \beta^2 = \alpha^2 \beta^2 + \alpha^2 \gamma^2 + \alpha^2 \delta^2 + \beta^2 \gamma^2 + \beta^2 \delta^2 + \gamma^2 \delta^2$$

$$= (\Sigma \alpha\beta)^2 - 2 \Sigma \alpha^2 \beta\gamma - 6 \alpha\beta\gamma\delta$$

$$= q^2 - 2 (pr - 4s) - 6s$$

$$= q^2 - 2pr + 2s.$$

$$\Sigma \alpha^3 \beta = (\Sigma \alpha^2) (\Sigma \alpha\beta) - \Sigma \alpha^2 \beta\gamma$$

$$= (p^2 - 2q) q - (pr - 4s)$$

$$= p^2q - 2q^2 - pr + 4s.$$

$$\Sigma \alpha^4 = (\Sigma \alpha^2)^2 - 2 \Sigma \alpha^2 \beta^2$$

$$= (p^2 - 2q)^2 - 2 (q^2 - 2pr + 2s)$$

$$= p^4 - 4p^2q + 2q^2 + 4pr - 4s.$$

Example 3. If α, β, γ are the roots of the equation $x^3 + ax^2 + bx + c = 0$, from the equation whose roots are $\alpha\beta$, $\beta\gamma$, and $\gamma\alpha$.

Solution.

The relations between the roots and coefficients are

$$\alpha + \beta + \gamma = -a$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = b$$

$$\alpha\beta\gamma = -c.$$

The required equation is

$$(x - \alpha\beta)(x - \beta\gamma)(x - \gamma\alpha) = 0$$

i.e., $x^3 - x^2(\alpha\beta + \beta\gamma + \gamma\alpha)(\alpha^2\beta\gamma + \alpha\beta^2\gamma + \alpha\beta\gamma^2)\, x - \alpha^2\beta^2\gamma^2 = 0$

i.e., $x^3 - x^2(\alpha\beta + \beta\gamma + \gamma\alpha) + x\,\alpha\beta\gamma(\alpha + \beta + \gamma) - (\alpha\beta\gamma)^2 = 0$

i.e., $x^3 - bx^2 + acx - c^2 = 0$

Example 4. If α, β, γ are the roots of the equation $x^3 + px^2 + qx + r = 0$, from the equation whose roots are $\beta + \gamma - 2\alpha$, $\gamma + \alpha - 2\beta$, $\alpha + \beta - 2\gamma$.

Solution.

We have $\alpha + \beta + \gamma = -p$

$$\alpha\beta + \beta\gamma + \gamma\alpha = q$$

$$\alpha\beta\gamma = -r.$$

In the required equation

S_1 = Sum of the roots $= \beta + \gamma - 2\alpha + \gamma + \alpha - 2\beta + \alpha + \beta - 2\gamma$

$= 0.$

S_2 = Sum of the products of the roots taken two at a time

$= (\beta + \gamma - 2\alpha)(\gamma + \alpha - 2\beta) + (\beta + \gamma - 2\alpha)(\alpha + \beta - 2\gamma) + (\alpha + \beta - 2\gamma)(\gamma + \alpha - 2\beta)$

$= (\alpha + \beta + \gamma - 3\alpha)(\alpha + \beta + \gamma - 3\beta) + 2$ similar terms

$= (-p - 3\alpha)(-p - 3\beta) + (-p - 3\alpha)(-p - 3\gamma) + (-p - 3\gamma)(-p - 3\beta)$

$= (p + 3\alpha)(p + 3\beta) + (p + 3\alpha)(p + 3\gamma) + (p + 3\gamma)(p + 3\beta)$

$= 3p^2 + 6p(\alpha + \beta + \gamma) + 9(\alpha\beta + \beta\gamma + \gamma\alpha)$

$= 3p^2 + 6p(-p) + 9q$

$= 9q - 3p^2.$

S_3 = Products of the roots

$= (\beta + \gamma - 2\alpha)(\gamma + \alpha - 2\beta)(\alpha + \beta - 2\gamma)$

$= (\alpha + \beta + \gamma - 3\alpha)(\alpha + \beta + \gamma - 3\beta)(\alpha + \beta + \gamma - 3\gamma)$

$= (-p - 3\alpha)(-p - 3\beta)(-p - 3\gamma)$

$= -\{ p^3 + 3p^2(\alpha + \beta + \gamma) + 9p(\alpha\beta + \beta\gamma + \gamma\alpha) + 27\alpha\beta\gamma \}$

$= -\{ p^3 + 3p^2(-p) + 9pq - 27r \}$

$= 2p^2 - 9pq + 27r$

Hence the required equation is

$x^3 - S_1x^2 + S_2x - S_3 = 0$

i.e., $x^3 + (9q - 3p^2)x - (2p^3 - 9pq + 27r) = 0.$

Example 5. If α, β, γ are the roots of the equation $x^3 + px^2 + qx + r = 0$ prove that

(1) $(\alpha + \beta)(\beta + \gamma)(\gamma + \alpha) = r - pq$

(2) $\alpha^3 + \beta^3 + \gamma^3 = -p^3 + 3pq - 3r.$

Solution.

We have $\alpha + \beta + \gamma = -p$

$\alpha\beta + \beta\gamma + \gamma\alpha = q$

$\alpha\beta\gamma = -r.$

(1). $(\alpha + \beta)(\beta + \gamma)(\gamma + \alpha) = [-(p + \alpha)(p + \beta)(p + \gamma)]$

Since $\alpha + \beta + \gamma = -p$ $\therefore \alpha + \beta = -p - \gamma$

$$= -[p^3 + p^2(\alpha + \beta + \gamma) + p(\alpha\beta + \beta\gamma + \gamma\alpha) + \alpha\beta\gamma]$$

$$= -[p^3 + p^2 \times -p + pq - r] = -[p^3 - p^3 + pq - r] = r - pq.$$

(2). $\alpha^3 + \beta^3 + \gamma^3 - 3\alpha\beta\gamma = (\alpha + \beta + \gamma)[\alpha^2 + \beta^2 + \gamma^2 - (\alpha\beta + \beta\gamma + \gamma\alpha)]$

$\sum \alpha^3 = \sum \alpha [\sum \alpha^2 - \sum \alpha\beta] + 3\alpha\beta\gamma;$

But $\sum \alpha^2 = (\sum \alpha)^2 - 2\sum \alpha\beta$

Therefore $\sum \alpha^3 = \sum \alpha [(\sum \alpha)^2 - 3\sum \alpha\beta] + 3\alpha\beta\gamma; = -p[p2 - 3q] - 3r = -p3 + 3pq - 3r.$

Example 6. If α, β, γ are the roots of the equation $x^3 + qx + r = 0$ find the values of

(1) $\sum \frac{1}{\beta+\gamma}$.

(2) $\sum \frac{\beta^2+\gamma^2}{\beta+\gamma}$

Solution.

Since α, β, γ are the roots of the equation $x^3 + qx + r = 0$.

We have $\alpha + \beta + \gamma = 0$

$\alpha\beta + \beta\gamma + \gamma\alpha = q$

$\alpha\beta\gamma = -r.$

Therefore $\beta + \gamma = -\alpha$

(1). $\sum \frac{1}{\beta+\gamma} = \sum \frac{1}{-\alpha} = -\left[\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}\right] = -\frac{\sum \alpha\beta}{\alpha\beta\gamma} = \frac{-q}{-r} = \frac{q}{r}$

(2). $\sum \frac{\beta^2+\gamma^2}{\beta+\gamma} = \sum \frac{(\beta+\gamma)^2 - 2\beta\gamma}{\beta+\gamma} = \frac{\sum\left[\alpha^2 + 2\frac{r}{\alpha}\right]}{-\alpha} = \frac{\sum \alpha^3 + 2r}{-\alpha^2} = -\sum \alpha - 2\sum \frac{r}{\alpha^2}$

$= -2r\sum \frac{1}{\alpha^2}$; since $\sum \alpha = 0$

But $\sum \frac{1}{\alpha^2} = \frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} = \frac{\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2}{\alpha^2\beta^2\gamma^2} = \frac{(\sum \alpha\beta)^2}{(\alpha\beta\gamma)^2}$ since $(\alpha\beta + \beta\gamma + \gamma\alpha)^2 = \sum \alpha^2\beta^2 + 2\alpha\beta\gamma\sum \alpha = \sum \alpha^2\beta^2$; since $\sum \alpha = 0$

$\sum \alpha^2\beta^2 = q^2$;$\sum \frac{1}{\alpha^2} = \frac{q^2}{r^2} = \frac{q^2}{r^2}$

$\therefore \sum \frac{\beta^2+\gamma^2}{\beta+\gamma} = \frac{-2q^2r}{r^2} = \frac{-2q^2}{r}$.

Example 7. If α, β, γ are the roots of the equation $x^3 - px^2 + qx - r = 0$ find the value of

(1). $\sum \frac{\beta^2+\gamma^2}{\beta\gamma}$

(2). $\sum(\beta + \gamma - \alpha)^2$.

Solution.

Since α, β, γ are the roots of the equation $x^3 - px^2 + qx - r = 0$

We have $\alpha + \beta + \gamma = p$

$$\alpha\beta + \beta\gamma + \gamma\alpha = q$$

$$\alpha\beta\gamma = r.$$

(1). $\sum \frac{\beta^2+\gamma^2}{\beta\gamma} = \frac{\beta^2+\gamma^2}{\beta\gamma} + \frac{\alpha^2+\beta^2}{\alpha\beta} + \frac{\alpha^2+\gamma^2}{\alpha\gamma} = \frac{\alpha(\beta^2+\gamma^2)+\gamma(\alpha^2+\beta^2)+\beta(\alpha^2+\gamma^2)}{\alpha\beta\gamma}$

$= \frac{\sum \alpha^2\beta}{\alpha\beta\gamma}$

But $\sum \alpha^2\beta = (\alpha\beta + \beta\gamma + \gamma\alpha)(\alpha + \beta + \gamma) - 3\alpha\beta\gamma$

$\frac{\sum \alpha^2\beta}{\alpha\beta\gamma} = \frac{(\alpha\beta + \beta\gamma + \gamma\alpha)(\alpha + \beta + \gamma) - 3\alpha\beta\gamma}{\alpha\beta\gamma} = \frac{qp-3r}{r}$.

(2). $\sum(\beta + \gamma - \alpha)^2 = \sum(\alpha + \beta + \gamma - 2\alpha)^2 = \sum(p - 2\alpha)^2 = \sum(p^2 + 4\alpha^2 - 4\alpha\beta)$

$= 3p^2 + 4\sum \alpha^2 - 4p\sum \alpha\beta$

$= 3p^2 + 4\left[\left(\sum \alpha\right)^2 - 2\sum \alpha\beta\right] - 4p^2$

$= 3p^2 + 4p^2 - 8q - 4p^2$

$= 3p^2 - 8q$.

Example 8. If α, β, γ are the roots of the equation $ax^3 + bx^2 + cx + d = 0$find the value of

$$\Sigma \frac{1}{\alpha^2\beta^2}$$

Solution.

Since α, β, γ are the roots of the equation $ax^3 + bx^2 + cx + d = 0$

We have $\alpha + \beta + \gamma = \frac{-b}{a}$

$$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a}$$

$$\alpha\beta\gamma = \frac{-d}{a}$$

$$\Sigma \frac{1}{\alpha^2\beta^2} = \frac{1}{\alpha^2\beta^2} + \frac{1}{\beta^2\gamma^2} + \frac{1}{\gamma^2\alpha^2} = \frac{\alpha^2+\beta^2+\gamma^2}{\alpha^2\beta^2\gamma^2} = \frac{(\alpha+\beta+\gamma)^2-2(\alpha\beta+\beta\gamma+\gamma\alpha)}{(\alpha\beta\gamma)^2} = \frac{\left(-\frac{b}{a}\right)^2-2\left(\frac{c}{a}\right)}{\left(\frac{d}{a}\right)^2}$$

$$= \frac{b^2-2ac}{d^2}$$